

On functions K and E generated by a sequence of moments

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Abstract We study the asymptotic behaviour of the entire function

$$E(z) = \sum_{n \geq 0} \frac{z^n}{\gamma(n+1)}$$

and the analytic function

$$K(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-s} \gamma(s) \, ds,$$

which naturally appear in various classical problems of analysis.

1 Introduction and main results

1.1 The functions K and E . In this work we study the asymptotic behavior of two analytic functions K and E generated by a sequence of moments $(\gamma(n+1))_{n \geq 0}$, where $\gamma(s)$ is an analytic function in the angle $\{s : |\arg(s+c)| < \alpha_0\}$ with $\frac{\pi}{2} < \alpha_0 \leq \pi$ and $c = c_\gamma > 0$. The function γ satisfies certain regularity properties, which we will list shortly. Here, we will only mention that $(\gamma(n))$ is a fastly growing sequence of positive numbers (so that $\lim_{n \rightarrow \infty} \gamma(n)^{1/n} = \infty$), and that, for some $\alpha \in (\frac{\pi}{2}, \alpha_0)$,

$$\lim_{\rho \rightarrow \infty} \frac{\log |\gamma(\rho e^{\pm i\alpha})|}{\rho} = -\infty.$$

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This allows us to define the functions

$$K(z) = \frac{1}{2\pi i} \int_{\mathcal{L}_\alpha} z^{-s} \gamma(s) ds, \quad (1.1.1)$$

where $\mathcal{L}_\alpha = \{z : |\arg(z)| = \alpha\}$ is a union of two rays traversed in such a way that $\operatorname{Im}(z)$ increases along \mathcal{L}_α (see Figure 1), and

$$E(z) = \sum_{n \geq 0} \frac{z^n}{\gamma(n+1)}. \quad (1.1.2)$$

The function K is analytic on the Riemann surface of $\log z$ (that is, the function $K(e^w)$

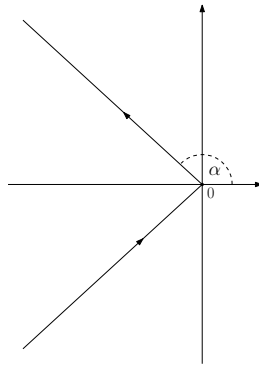


Fig. 1: Contour \mathcal{L}_α

is entire), while the function E is an entire one.

The assumptions on the function γ , which we will impose shortly, will allow us, moving the integration contour, represent the function $K(t)$ for $t \geq 0$ as the inverse Mellin transform of γ :

$$K(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} \gamma(s) ds, \quad c > 0, \quad (1.1.3)$$

where the integral does not depend on the choice of $c > 0$. Then, by the inversion formula for the Mellin transform, K solves the moment problem

$$\int_0^\infty t^n K(t) dt = \gamma(n+1), \quad n \in \mathbb{Z}_+.$$

The functions K and E naturally appear in various classical problems of analysis, for instance, in the Borel-type moment summation of divergent series [10] and in studying in convergence of certain interpolation problems for entire functions [6, 7, 8]. It is also worth mentioning that Beurling [3, 4] singled out a class of functions γ for which the function K is positive on the positive half-line (see Section 1.7 and Appendix B). Then, our Theorem 1 gives explicit asymptotics of solutions to a large class of determinate Stieltjes moment problem.

Our interest originated in Beurling's approach to the problems of description of the Taylor coefficients and of summation of the divergent Taylor series in various classes of smooth functions [3, 4, 12].

It could be that the results presented here are known to experts. On the other hand, we were unable to locate them in the literature and we believe that they are of certain interest.

At last, we note that juxtaposing the right-hand sides of (1.1.1) and (1.1.2), we may expect some match between the growth of E and the decay of K on the positive half-line and nearby. In the prototype case when γ is Euler's gamma-function, $\gamma(n+1) = n!$, $E(z) = e^z$, $K(z) = e^{-z}$, and this match is perfect.

1.2 Admissible functions. From now on, we will assume that the function γ is analytic and non-vanishing in the angle $\{s : |\arg(s+c)| < \alpha_0\}$ with $\frac{\pi}{2} < \alpha_0 \leq \pi$ and $c = c_\gamma > 0$, and is positive on $(-c, +\infty)$. We put

$$L(s) = \gamma(s)^{1/s} \quad \text{and} \quad \varepsilon(s) = s \frac{L'(s)}{L(s)}.$$

Definition. We call the function γ admissible, if the function ε is positive and bounded on \mathbb{R}_+ , and satisfies the following conditions:

$$(A) \quad \int_0^\infty \frac{\varepsilon(\rho)}{\rho} d\rho = \infty,$$

$$(B) \quad \rho|\varepsilon'(\rho)| = o(\varepsilon(\rho)) \text{ as } \rho \rightarrow \infty,$$

$$(C) \quad \text{for } s = \rho e^{i\theta}, \rho \rightarrow \infty, \text{ one has } \varepsilon(s) = (1 + o(1))\varepsilon(\rho), \text{ uniformly in any angle } |\theta| \leq \alpha_0 - \delta.$$

Condition (A) means that the function L is unbounded. Condition (B) says that the function ε is slowly varying. Everywhere below, we always assume that the function γ is admissible.

It is not difficult to see that conditions (B) and (C) yield that, for $s = \rho e^{i\theta}$, $\rho \rightarrow \infty$,

$$(D) \quad \log L(\rho e^{i\theta}) = \int_0^\rho \frac{\varepsilon(u)}{u} du + i\theta\varepsilon(\rho) + o(\varepsilon(\rho)),$$

$$(E) \quad s\varepsilon'(s) = o(\varepsilon(\rho)),$$

also uniformly in any angle $|\theta| \leq \alpha_0 - \delta$.

Indeed, condition (D) follows from (C) by integration, while (E) follows from (C) and (B) due to the analyticity of ε .

Below, in Sections 1.5 and 1.6, we will give several examples and constructions of admissible functions γ .

1.3 The saddle-point equation. It is clear, at least intuitively, that the asymptotic behavior of the functions $K(z)$ and $E(z)$ for large z should be determined by the saddle-point of the function $s \mapsto \log \gamma(s) - s \log z = s \log L(s) - s \log z$, that is by the equation

$$\log L(s) + s \frac{L'(s)}{L(s)} = \log z, \quad (1.3.1)$$

which we will call *the saddle-point equation*.

For $0 < \alpha < \alpha_0$ and $\rho_0 > 0$, put

$$S(\alpha, \rho_0) = \{s : |\arg(s)| < \alpha, |s| > \rho_0\}$$

and let

$$\Phi(s) = \log L(s) + s \frac{L'(s)}{L(s)}.$$

Note that for $s \in S(\alpha, \rho_0)$ and $|s| = \rho$ sufficiently large, by (B),(C) and (E), we have

$$\operatorname{Re} \Phi'(s) = (1 + o(1)) \frac{\varepsilon(\rho)}{\rho} > 0.$$

Thus, for $s_1, s_2 \in S(\alpha, \rho_0)$, $s_1 \neq s_2$,

$$\operatorname{Re} \frac{\Phi(s_2) - \Phi(s_1)}{s_2 - s_1} = \int_0^1 \operatorname{Re} \Phi'(s_1 + t(s_2 - s_1)) dt > 0,$$

provided that ρ_0 is large enough. Therefore, for ρ_0 sufficiently large, the function Φ , that is, the LHS of the saddle-point equation, is a univalent function in $S(\alpha, \rho_0)$. From here on, we assume that this is the case. Then, we put

$$\Omega(\alpha, \rho_0) = \{\log z = \Phi(s) : s \in S(\alpha, \rho_0)\}.$$

This is a domain on the Riemann surface of $\log z$. If the index ρ_0 is not essential, we will skip it, to simplify notation.

Note that if $\rho = |s|$ is sufficiently large, by (C) and (D), we have

$$\operatorname{Im} \Phi(s) = (\theta + o(1)) \varepsilon(\rho), \quad s = \rho e^{i\theta}.$$

Thus, choosing ρ_0 sufficiently large, we can treat $\Omega(\alpha, \rho_0)$ as a subdomain of the slit plane $\mathbb{C} \setminus \mathbb{R}_-$, provided that

$$\limsup_{\rho \rightarrow \infty} \varepsilon(\rho) < \frac{\pi}{\alpha},$$

in particular, whenever $\varepsilon(\rho) = o(1)$, as $\rho \rightarrow \infty$.

In what follows, by $s_z = \rho_z e^{i\theta_z}$ we always denote the unique solution of the saddle-point equation (1.3.1).

1.4 Asymptotics of the functions K and E . We are now able to present our results.

Theorem 1. *Suppose that the function γ is admissible. Then, for any $\delta > 0$, we have*

$$K(z) = (1 + o(1)) \sqrt{\frac{L(s)}{2\pi L'(s)}} \exp\left(-s^2 \frac{L'(s)}{L(s)}\right), \quad z \rightarrow \infty$$

uniformly in $\Omega(\alpha_0 - \delta)$. Here $s = s_z$ and the branch of the square root is positive on the positive half-line.

Theorem 2. *Suppose that the function γ is admissible and that*

$$\limsup_{\rho \rightarrow \infty} \varepsilon(\rho) < 2. \quad (1.4.1)$$

Then, given a sufficiently small $\delta > 0$, we have

$$zE(z) + \frac{1}{\gamma(0)} = (1 + o(1)) \sqrt{2\pi \frac{L(s)}{L'(s)}} \exp\left(s^2 \frac{L'(s)}{L(s)}\right) + o(1), \quad z \rightarrow \infty,$$

uniformly in $\Omega(\pi/2 + \delta)$, and

$$zE(z) + \frac{1}{\gamma(0)} = o(1), \quad z \rightarrow \infty$$

uniformly in $\mathbb{C} \setminus \Omega(\pi/2 + \delta)$. Here, also $s = s_z$ and the branch of the square root is positive on the positive half-line.

Note that it is not difficult to drop assumption (1.4.1) in Theorem 2 at the expense of a more complicated conclusion. We shall not do this here. One of the reasons is that we are mainly interested in the case when $\varepsilon(\rho) = o(1)$ as $\rho \rightarrow \infty$.

We also note that the asymptotics given in Theorems 2 and 1 are known in the case when there exists a positive limit

$$\lambda = \lim_{r \rightarrow \infty} \varepsilon(\rho),$$

cf. [9, 13]. In this case, E is an entire function of order $\frac{1}{\lambda}$. The logarithmic case $L(s) = \log(s + e)$ is also classical and goes back to Lindelöf.

1.4.1 An example. $L(s) = \log^\beta(s + e)$, $\beta > 0$. In this case, the saddle-point equation (1.3.1) has the form

$$\beta \log \log(s + e) + \frac{\beta s}{s + e} \cdot \frac{1}{\log(s + e)} = \log z,$$

which readily simplifies to

$$\log \log s + \frac{1}{\log s} + O\left(\frac{1}{s \log s}\right) = \frac{1}{\beta} \log z,$$

whence

$$s = \left(1 - \frac{1 + o(1)}{2} z^{-1/\beta}\right) \exp(z^{1/\beta} - 1).$$

Then

$$K(z) = (1 + o(1)) \sqrt{\frac{s \log s}{2\pi\beta}} \exp\left(-\beta \frac{s}{\log s}\right)$$

uniformly in any domain $\Omega(\pi - \delta)$, and

$$zE(z) + 1 = (1 + o(1)) \sqrt{\frac{2\pi}{\beta} s \log s} \exp\left(\beta \frac{s}{\log s}\right) + o(1)$$

uniformly in $\Omega(\pi/2 + \delta)$ with sufficiently small $\delta > 0$.

We note that the entire function E has nearly maximal growth in the curvilinear strip $\Omega(\pi/2)$, while the analytic function K has nearly fastest decay in $\Omega(\pi/2)$, and that, for sufficiently large r_0 ,

$$\Omega(\pi/2) \cap \{z : |z| > r_0\} = \{z = re^{i\psi} : |\psi| \leq \Psi(r), r > r_0\},$$

where

$$\Psi(r) = \frac{\pi\beta}{2} \left(r^{-1/\beta} + \left(\frac{\pi^2}{8} - \frac{1}{2} \right) r^{-3/\beta} + O(r^{-4/\beta}) \right), \quad r \rightarrow \infty.$$

1.4.2 The observation we have just made is quite general, For any curvilinear semistrip Ω which is bounded by two sufficiently regularly varying curves $\{z = re^{\pm i\Psi(r)}\}$, one can find a function γ , satisfying our regularity conditions (A), (B) and (C), so that the entire function E will have nearly maximal growth in Ω , while the analytic function K will have nearly fastest decay in Ω . We shall not pursue that matter here.

1.5 Examples of admissible functions γ . We start with several straightforward observations:

1.5.1 The shifted Euler's Gamma-function $\Gamma(s + c)$, $c > 0$, is admissible.

1.5.2 If the function γ is admissible, then the functions

$$s \mapsto \frac{\gamma(s + c)}{\gamma(c)}, \quad c > 0, \quad \text{and} \quad s \mapsto \gamma(s)e^{\tau s}, \quad \tau \in \mathbb{R},$$

are also admissible.

1.5.3 Denote by \log_k the k -th iterate of the logarithmic function. Then the function

$$\gamma(s) = \exp \left[as \log_k^b(s + c_k) \right], \quad k \in \mathbb{N},$$

is admissible provided that $a > 0$, $b > 0$ (and $b \leq 1$ for $k = 1$), and that $c_k > 0$ are sufficiently large.

The following simple rules allow one to construct new admissible functions from the given ones:

1.5.4 If γ is admissible and $a > 0$, then the function γ^a is also admissible.

1.5.5 If γ_1 and γ_2 are admissible, then $\gamma_1 \cdot \gamma_2$ is always admissible, while $\frac{\gamma_1}{\gamma_2}$ is admissible provided that $\gamma_1 \geq \gamma_2$ on $(0, \infty)$ and that the function

$$\rho \mapsto \left(\frac{\gamma_1(\rho)}{\gamma_2(\rho)} \right)^{1/\rho}, \quad \rho > 0,$$

is non-decreasing and unbounded.

1.5.6 If $\gamma(s) = L(s)^s$ is admissible, then the function

$$s \mapsto (\log L(s+1))^s$$

is admissible as well.

1.6 Admissible functions with prescribed asymptotic behavior. It is not difficult to construct admissible functions with prescribed asymptotic behavior on the positive ray. The next result is a version of the known observation that if h is a slowly varying function on $[0, \infty)$ (that is, $\rho h'(\rho) = o(h(\rho))$ as $\rho \rightarrow \infty$), then the function

$$\mathfrak{h}(s) = s \int_0^\infty \frac{h(u) du}{(u+s)^2}$$

is analytic in $\{s : |\arg(s)| < \pi\}$, slowly varying on \mathbb{R}_+ , and for $\rho \rightarrow \infty$ satisfies $\mathfrak{h}(\rho e^{i\theta}) = (1 + o(1))h(\rho)$ uniformly in any angle $|\arg(s)| \leq \pi - \delta$.

Theorem 3. Suppose $\ell : [0, \infty) \rightarrow (0, \infty)$ is an unboundedly increasing C^1 -function such that the function

$$\rho \mapsto \rho \frac{\ell'(\rho)}{\ell(\rho)}$$

is slowly varying and bounded for $\rho > 0$. Then, for any $c > 0$, the function

$$\gamma(s) = \exp \left(s^2 \int_c^\infty \frac{\ell'(u)}{\ell(u)} \frac{du}{s+u} \right), \quad |\arg(s+c)| < \pi, \quad (1.6.1)$$

is admissible and

$$\lim_{\rho \rightarrow \infty} \frac{\log \gamma(\rho)}{\rho \log \ell(\rho)} = 1.$$

If, in addition, there exists the limit, $\lim_{\rho \rightarrow \infty} \rho \frac{\ell'(\rho)}{\ell(\rho)}$, then

$$\lim_{\rho \rightarrow \infty} \frac{\ell(\rho)}{\gamma(\rho)^{1/\rho}} = \ell(0).$$

For the reader's convenience, we give the proof of Theorem 3 in Appendix A.

1.7 Admissible functions of positive type. Beurling observed in [3, 4] that analytic functions that admit integral representations similar to (1.6.1) have special positivity properties which yield that $K(t) \geq 0$ for $t > 0$. This provides us with a large class of explicit integral representations for solutions $K(t)$ to the Stieltjes moment problem with known asymptotics as $t \rightarrow \infty$ given by Theorem 1. We shall discuss this in Appendix B.

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2 Preliminaries

Put

$$G(z, s) := \log \gamma(s) - s \log z = s \log L(s) - s \log z.$$

Then

$$K(z) = \frac{1}{2\pi i} \int_{\mathcal{L}_\alpha} e^{G(z, s)} ds, \quad (2.0.1)$$

where \mathcal{L}_α is the same contour as on Figure 1, and

$$zE(z) + \frac{1}{\gamma(0)} = \int_{-\sigma_0}^{\infty} e^{-G(z, s)} ds + o(1). \quad (2.0.2)$$

The latter relation easily follows from the classical Abel-Plana summation formula (see Section 4). In this section, we collect estimates of the function G and its derivatives needed for the asymptotic estimates of the integrals on the RHS of (2.0.1) and (2.0.2).

Recall that, for any $0 < \alpha < \alpha_0$, the function $\Phi(s) = \log L(s) + \varepsilon(s)$ (where, as before, $\varepsilon(s) = s \frac{L'}{L}(s)$) is univalent in the domain $S(\alpha, \rho_0) = \{s: |\arg(s)| < \alpha, |s| > \rho_0\}$ with sufficiently large ρ_0 , and that we denote $\Omega(\alpha, \rho_0) = \Phi(S(\alpha, \rho_0))$. Hence, for any $z \in \Omega(\alpha, \rho_0)$, the function $s \mapsto G(z, s)$ has a unique critical point, which we denote by $s_z = \rho_z e^{i\theta_z}$, and this is the unique saddle-point of the function $s \mapsto \operatorname{Re} G(z, s)$.

2.1 Derivatives of G . The first derivative $G'_s(z, s)$ equals $\log L(s) + \varepsilon(s) - \log z$. Recalling conditions (D) and (C), and the saddle-point equation

$$\begin{aligned} \log z &= \log L(s_z) + \varepsilon(s_z) \\ &= \int_0^{\rho_z} \frac{\varepsilon(u)}{u} du + \varepsilon(\rho_z) + i\theta_z \varepsilon(\rho_z) + o(\varepsilon(\rho_z)), \end{aligned}$$

we get

$$G'_s(z, \rho e^{i\theta}) = \int_{\rho_z}^{\rho} \frac{\varepsilon(u)}{u} du + i\varepsilon(\rho_z)(\theta - \theta_z) + \chi_1(z) + \chi_2(\rho e^{i\theta}), \quad (2.1.1)$$

where $\chi_1(z) = o(\varepsilon(\rho_z))$ uniformly in $z \in \bar{\Omega}(\alpha_0 - \delta)$, $z \rightarrow \infty$, and $\chi_2(\rho e^{i\theta}) = o(\varepsilon(\rho))$ uniformly in $|\theta| \leq \alpha_0 - \delta$, $\rho \rightarrow \infty$.

The second derivative $G''_{ss}(z, s)$ does not depend on z and equals

$$\begin{aligned} G''_{ss}(z, s) &= \frac{L'}{L}(s) + \varepsilon'(s) \\ &= (1 + o(1)) \frac{\varepsilon(\rho)}{s} \quad (\text{by (B) and (C)}) \end{aligned} \quad (2.1.2)$$

uniformly in any angle $|\arg(s)| \leq \alpha_0 - \delta$. In particular,

$$G''_{ss}(z, s_z) = (1 + o(1)) \frac{\varepsilon(\rho_z)}{\rho_z} e^{-i\theta_z}. \quad (2.1.3)$$

Since the function $\varepsilon(\rho)$ is slowly varying, we see that if we will succeed to correctly deform the integration contours in the integrals on the RHSs of (2.0.1) and (2.0.2), then the asymptotics of these integrals will be determined by a neighbourhood of the saddle point s_z of size ρ_z^c with any $c > \frac{1}{2}$.

2.2 Behaviour of G in a neighbourhood of the saddle point s_z . We fix a small positive $\delta_1 < \frac{1}{2}$ (for instance, the value $\delta_1 = \frac{1}{8}$ will suffice for our purposes) and assume that $|s - s_z| \leq \rho_z^{1-\delta_1}$. By (2.1.2), combined with condition (B), we have

$$G''_{ss}(z, s) = (1 + o(1)) G''_{ss}(z, s_z) \quad (2.2.1)$$

uniformly in $|s - s_z| \leq \rho_z^{1-\delta_1}$, whence,

$$G(z, s) = G(z, s_z) + \left(\frac{1}{2} + o(1)\right) (s - s_z)^2 \frac{\varepsilon(\rho_z)}{\rho_z} e^{-i\theta_z} \quad (2.2.2)$$

also uniformly in $|s - s_z| \leq \rho_z^{1-\delta_1}$, $z \in \bar{\Omega}(\alpha_0 - \delta)$, $z \rightarrow \infty$. Thus,

- the function $w \mapsto \operatorname{Re} G(z, s_z + w)$ has the fastest decay in the directions $w = \pm i e^{i\theta_z/2}$

and

- the fastest growth in the directions $w = \pm e^{i\theta_z/2}$.

Let Γ be a smooth simple curve that traverses once the disk

$$D(s_z) = \{|s - s_z| \leq \rho_z^{1-\delta_1}\}$$

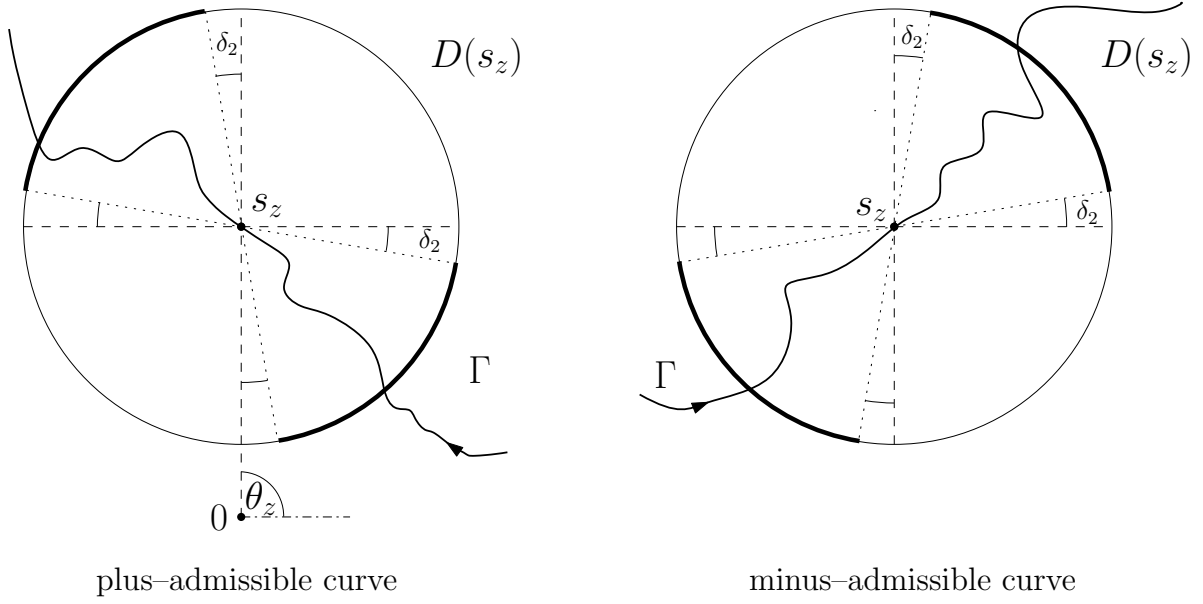
and passes through the saddle-point s_z . We call the curve Γ *plus-admissible* if it

- enters $D(s_z)$ through the arc $\{s = s_z - i\rho_z^{1-\delta_1} e^{i\varphi} : |\varphi - \frac{1}{2}\theta_z| \leq \frac{1}{4}\pi - \delta_2\}$,
- exists $D(s_z)$ through the arc $\{s = s_z + i\rho_z^{1-\delta_1} e^{i\varphi} : |\varphi - \frac{1}{2}\theta_z| \leq \frac{1}{4}\pi - \delta_2\}$,

and

- $\Gamma \cap D(s_z)$ does not leave the set

$$\{s = s_z + ite^{i\varphi} : -\rho_z^{1-\delta_1} \leq t \leq \rho_z^{1-\delta_1}, |\varphi - \frac{1}{2}\theta_z| \leq \frac{1}{4}\pi - \delta_2\}.$$



Similarly, we say that Γ is *minus-admissible*, if

- it enters the disk $D(s_z)$ through the arc $\{s = s_z - \rho_z^{1-\delta_1} e^{i\varphi} : |\varphi - \frac{1}{2}\theta_z| \leq \frac{1}{4}\pi - \delta_2\}$,
- exits $D(s_z)$ through the arc $\{s = s_z + \rho_z^{1-\delta_1} e^{i\varphi} : |\varphi - \frac{1}{2}\theta_z| \leq \frac{1}{4}\pi - \delta_2\}$,

and

- $\Gamma \cap D(s_z)$ does not leave the set

$$\{s = s_z + te^{i\varphi} : -\rho_z^{1-\delta_1} \leq t \leq \rho_z^{1-\delta_1}, |\varphi - \frac{1}{2}\theta_z| \leq \frac{1}{4}\pi - \delta_2\}.$$

Lemma 2.2.1. *Suppose that the curve Γ is plus-admissible. Then*

$$\int_{\Gamma \cap D(s_z)} e^{G(z,s)} ds = (i + o(1)) \sqrt{2\pi \frac{L}{L'}(s_z)} e^{-s_z^2 \frac{L'}{L}(s_z)}.$$

If the curve Γ is minus-admissible, then

$$\int_{\Gamma \cap D(s_z)} e^{-G(z,s)} ds = (1 + o(1)) \sqrt{2\pi \frac{L}{L'}(s_z)} e^{s_z^2 \frac{L'}{L}(s_z)}.$$

Both asymptotic relations hold uniformly in $z \in \bar{\Omega}(\alpha_0 - \delta)$, $z \rightarrow \infty$. The branch of the square root on the RHSs is positive when z belongs to the positive ray.

Note that $s_z^2 L'(s_z)/L(s_z) = s_z \varepsilon(s_z)$, and $L(s_z)/L'(s_z) = s_z/\varepsilon(s_z) = (1 + o(1))s_z/\varepsilon(\rho_z)$.

Proof of Lemma 2.2.1: We prove only the first statement; the proof of the second one is very similar. We start with the special case, when $\Gamma \cap D(s_z)$ is the segment I of the fastest decay of the function $w \mapsto \operatorname{Re} G(z, s_z + w)$:

$$I = \{s = s_z + ite^{i\theta_z/2} : -\rho_z^{1-\delta_1} \leq t \leq \rho_z^{1-\delta_1}\}.$$

In this case,

$$\begin{aligned} \int_I e^{G(z,s)} ds &\stackrel{(2.2.2)}{=} e^{G(z,s_z)} \int_{I_+} e^{(\frac{1}{2}+o(1))(s-s_z)^2 \frac{\varepsilon(\rho_z)}{\rho_z}} e^{-i\theta_z} ds \\ &= e^{G(z,s_z)} i e^{i\theta_z/2} \sqrt{\frac{\rho_z}{\varepsilon(\rho_z)}} \int_{-\rho_z^{1/2-\delta_1}\varepsilon(\rho_z)^{-1/2}}^{\rho_z^{1/2-\delta_1}\varepsilon(\rho_z)^{-1/2}} e^{-(\frac{1}{2}+o(1))t^2} dt. \end{aligned}$$

The function $\varepsilon(\rho)$ is slowly varying. Hence, for any $c > 0$, the function $\rho^{-c}\varepsilon(\rho)$ decays to 0 as $\rho \rightarrow \infty$. Since $\delta_1 < \frac{1}{2}$, we conclude that $\rho_z^{1/2-\delta_1}\varepsilon(\rho_z)^{-1/2} \rightarrow +\infty$ uniformly in $z \in \bar{\Omega}(\alpha_0 - \delta)$, $z \rightarrow \infty$, and therefore, the integral on the RHS converges to $\sqrt{2\pi}$ also uniformly.

It remains to note that $G(z, s_z) = -s_z^2 \frac{L'}{L}(s_z)$, and that, by condition (C),

$$e^{-i\theta_z/2} \sqrt{\frac{\rho_z}{\varepsilon(\rho_z)}} = (1 + o(1)) \sqrt{\frac{L}{L'}(s_z)},$$

completing the proof of this special case of Lemma 2.2.1.

To move to the general case, we note that on the boundary circumference $|s - s_z| = \rho_z^{1-\delta_1}$ the function $\operatorname{Re} G(z, s)$ is much smaller than at the saddle-point s_z . More precisely, we claim that *given small positive δ , δ_1 , δ_2 , there exists a sufficiently large $\rho_0 = \rho_0(\delta, \delta_1, \delta_2)$ so that, for $z \in \bar{\Omega}(\alpha_0 - \delta, \rho_0)$ and $|\varphi - \frac{1}{2}(\theta_z \pm \pi)| \leq \frac{1}{4}\pi - \delta_2$, we have*

$$\operatorname{Re} G(z, s_z + \rho_z^{1-\delta_1} e^{i\varphi}) \leq \operatorname{Re} G(z, s_z) - \rho_z^{1-3\delta_1}.$$

Indeed, for $|\varphi - \frac{1}{2}(\theta_z \pm \pi)| \leq \frac{1}{4}\pi - \delta_2$, we have $\cos(2\varphi - \theta_z) \leq -c < 0$, whence, taking into account (2.2.2),

$$\begin{aligned} \operatorname{Re} G(z, s) &\leq \operatorname{Re} G(z, s_z) - \left(\frac{1}{2} + o(1)\right) \rho_z^{2-2\delta_1} \frac{\varepsilon(\rho_z)}{\rho_z} \cdot c \\ &\leq \operatorname{Re} G(z, s_z) - c_1 \rho_z^{1-2\delta_1} \varepsilon(\rho_z), \end{aligned}$$

provided that ρ_z is sufficiently large. Since the function $\varepsilon(\rho)$ is slowly varying, for ρ sufficiently large, we have $c_1 \cdot \varepsilon(\rho) \geq \rho^{-\delta_1}$, proving the claim.

At last, $\rho_z^{1-3\delta_1} e^{-\rho_z^{1-3\delta_1}}$ is much smaller than $\sqrt{\frac{\rho_z}{\varepsilon(\rho_z)}}$. Thus, using Cauchy's theorem, we can replace the segment I by $\Gamma \cap D(s_z)$ for any plus-admissible curve Γ , completing the proof. \square

2.3 Asymptotics of $\operatorname{Re} G$. We have

$$\operatorname{Re} G(re^{i\psi}, \rho e^{i\theta}) = \operatorname{Re} [\rho e^{i\theta} \log L(\rho e^{i\theta}) - \rho e^{i\theta} (\log r + i\psi)]$$

$$\stackrel{(D)}{=} \rho \cos \theta \left(\int_0^\rho \frac{\varepsilon(u)}{u} du - \log r \right) - \rho \sin \theta (\theta \varepsilon(\rho) - \psi) + o(\rho \varepsilon(\rho))$$

uniformly in $|\theta| \leq \alpha_0 - \delta$, $\rho \rightarrow \infty$. Recalling the equation

$$\log z = \log L(s_z) + \varepsilon(s_z)$$

for the saddle point $s_z = \rho_z e^{i\theta_z}$ and using conditions (C) and (D), we see that

$$\begin{aligned} \operatorname{Re} G(z, \rho e^{i\theta}) &= \rho \cos \theta \left(\int_{\rho_z}^{\rho} \frac{\varepsilon(u)}{u} du - \varepsilon(\rho_z) \right) \\ &\quad - \rho \sin \theta (\theta \varepsilon(\rho) - \theta_z \varepsilon(\rho_z)) + \rho (\chi_1(z) + \chi_2(\rho e^{i\theta})), \end{aligned} \quad (2.3.1)$$

where, as before, $\chi_1(z) = o(\varepsilon(\rho_z))$ uniformly in $z \in \bar{\Omega}(\alpha_0 - \delta)$, $z \rightarrow \infty$, and $\chi_2(\rho e^{i\theta}) = o(\varepsilon(\rho))$ uniformly in $|\theta| \leq \alpha_0 - \delta$, $\rho \rightarrow \infty$.

2.4 Estimates of $\operatorname{Re} G$ on arcs of the circumference $|s| = \rho_z$.

Lemma 2.4.1. *Suppose that $\max(|\theta|, |\theta_z|) \leq \alpha_0 - \delta$. Then*

$$\operatorname{Re} G(z, \rho_z e^{i\theta}) - \operatorname{Re} G(z, \rho_z e^{i\theta_z}) = -(f(\theta, \theta_z) + o(1)) \rho_z \varepsilon(\rho_z), \quad (2.4.1)$$

where $f(\theta, \theta_z) = \cos \theta - \cos \theta_z + (\theta - \theta_z) \sin \theta$, and

$$\frac{\partial^2}{\partial \theta^2} \operatorname{Re} G(z, \rho_z e^{i\theta}) = (h(\theta) + o(1)) \rho_z \varepsilon(\rho_z), \quad (2.4.2)$$

where $h(\theta) = -\cos \theta + (\theta - \theta_z) \sin \theta$. Both estimates are uniform as $z \rightarrow \infty$.

Proof of Lemma 2.4.1: Estimate (2.4.1) immediately follows from asymptotics (2.3.1). The proof of (2.4.2) is also straightforward. We have

$$\frac{\partial^2}{\partial \theta^2} G(z, \rho_z e^{i\theta}) = -\rho_z e^{i\theta} G'_s(z, \rho_z e^{i\theta}) - \rho_z^2 e^{2i\theta} G''_{ss}(z, \rho_z e^{i\theta}),$$

$$G'_s(z, \rho_z e^{i\theta}) \stackrel{(2.1.1)}{=} i(\theta - \theta_z) \varepsilon(\rho_z) + o(\varepsilon(\rho_z)),$$

and

$$G''_{ss}(z, \rho_z e^{i\theta}) \stackrel{(2.1.2)}{=} (1 + o(1)) \frac{\varepsilon(\rho_z)}{\rho_z} e^{-i\theta_z}.$$

Therefore,

$$\begin{aligned} \frac{\partial^2}{\partial \theta^2} \operatorname{Re} G(z, \rho_z e^{i\theta}) &= \operatorname{Re} \left[\frac{\partial^2}{\partial \theta^2} G(z, \rho_z e^{i\theta}) \right] \\ &= -\rho_z \varepsilon(\rho_z) \operatorname{Re} \left[1 + e^{i\theta} (i(\theta - \theta_z) + o(1)) \right] \\ &= -\rho_z \varepsilon(\rho_z) (\cos \theta - (\theta - \theta_z) \sin \theta + o(1)), \end{aligned}$$

completing the proof. □

2.5 Estimate of $\operatorname{Re} G$ on segments that pass through the saddle point.

Lemma 2.5.1. *Let t be a real number such that $|t| \leq 1 - \delta_3$. Then,*

$$\frac{\partial^2}{\partial t^2} \operatorname{Re} G(z, s_z + t\rho_z e^{i\varphi}) = (1 + o(1)) \frac{\rho_z \varepsilon(\rho_z)}{|e^{i\theta_z} + te^{i\varphi}|} \cos(2\varphi - \arg(e^{i\theta_z} + te^{i\varphi})).$$

Proof of Lemma 2.5.1:

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \operatorname{Re} G(z, s_z + t\rho_z e^{i\varphi}) &= \operatorname{Re} \left[\rho_z^2 e^{2i\varphi} G''_{ss}(z, s_z + t\rho_z e^{i\varphi}) \right] \\ &\stackrel{(2.1.2)}{=} \operatorname{Re} \left[\rho_z^2 e^{2i\varphi} \cdot (1 + o(1)) \frac{\varepsilon(s_z + t\rho_z e^{i\varphi})}{s_z + t\rho_z e^{i\varphi}} \right] \end{aligned}$$

By (C) and (B), the RHS equals

$$(1 + o(1)) \frac{\rho_z \varepsilon(\rho_z)}{|e^{i\theta_z} + te^{i\varphi}|} \cos(2\varphi - \arg(e^{i\theta_z} + te^{i\varphi}))$$

which proves the lemma. \square

2.6 Tail estimates of $\operatorname{Re} G$.

Lemma 2.6.1. *The function $\rho \mapsto \frac{\partial \operatorname{Re} G(z, \rho e^{i\theta})}{\partial \rho}$, $\rho \geq \rho_z$, increases whenever $|\theta| \leq \frac{\pi}{2} - \delta_4$, and decays whenever $|\theta| \geq \frac{\pi}{2} + \delta_4$. Furthermore, for $\rho_z \leq \rho \leq 2\rho_z$,*

$$\frac{\partial \operatorname{Re} G(z, \rho e^{i\theta})}{\partial \rho} = \varepsilon(\rho_z) \left(\log \frac{L(\rho)}{L(\rho_z)} \cos \theta - (\theta - \theta_z) \sin \theta + o(1) \right)$$

uniformly in $z \in \bar{\Omega}(\alpha_0 - \delta)$, $z \rightarrow \infty$.

Proof of Lemma 2.6.1: The first claim follows since, due to the asymptotics (2.1.2), we have

$$\frac{\partial^2 \operatorname{Re} G(z, \rho e^{i\theta})}{\partial \rho^2} = (1 + o(1)) \frac{\varepsilon(\rho)}{\rho} \cos \theta.$$

To check the second claim we write

$$\frac{\partial \operatorname{Re} G(z, \rho e^{i\theta})}{\partial \rho} = \operatorname{Re} \left[e^{i\theta} G'_s(z, \rho e^{i\theta}) \right]$$

$$\stackrel{(2.1.1)}{=} \operatorname{Re} \left[e^{i\theta} \left(\int_{\rho_z}^{\rho} \frac{\varepsilon(u)}{u} du + i\varepsilon(\rho_z)(\theta - \theta_z) + o(\varepsilon(\rho_z)) \right) \right] \quad (\text{by (C) and (D)})$$

$$= \varepsilon(\rho_z) \operatorname{Re} \left[e^{i\theta} \left(\log \frac{L(\rho)}{L(\rho_z)} + i(\theta - \theta_z) + o(1) \right) \right]$$

$$= \varepsilon(\rho_z) \left(\log \frac{L(\rho)}{L(\rho_z)} \cos \theta - (\theta - \theta_z) \sin \theta + o(1) \right) \quad (\text{by (C)}),$$

proving the second claim. \square

2.7 Estimate of $\operatorname{Re} G$ in a bounded sector.

Lemma 2.7.1. *Given positive ρ_1 and δ , there exists a positive C so that*

$$\max\left\{\left|\operatorname{Re} G(z, \rho e^{i\theta}) + \operatorname{Re}(s) \int_0^{\rho z} \frac{\varepsilon(u)}{u} du\right| : \rho \leq \rho_1, |\theta| \leq \alpha - \delta\right\} \leq C,$$

uniformly in $z \in \bar{\Omega}(\alpha_0 - \delta)$.

Proof of Lemma 2.7.1: We have

$$\operatorname{Re} G(z, s) = \operatorname{Re}(s) \cdot \log r + O(1),$$

uniformly in s , $|s| \leq \rho_1$, $|\arg(s)| \leq \alpha_0 - \delta$, and in $z \in \bar{\Omega}(\alpha_0 - \delta)$. Recalling that, by the saddle-point equation,

$$\log r = \int_0^{\rho z} \frac{\varepsilon(u)}{u} du + O(1),$$

also uniformly in $z \in \bar{\Omega}(\alpha_0 - \delta)$, we get the result. \square

3 Proof of Theorem 1

We fix several sufficiently small positive parameters δ, δ_i , some of which already appeared in lemmas proven in the previous section. Some restrictions on these parameters will be imposed in the course of the proof. By c and C we denote various positive constants that may depend on these parameters, the values of these constants are inessential for our purposes and may differ from line to line.

In the course of the proof, all expressions that are

$$o(1) \sqrt{\frac{\rho z}{\varepsilon(\rho z)}} e^{\operatorname{Re} G(z, s_z)}$$

will be called *negligible*.

Without loss of generality, we assume during the proof that the saddle point s_z lies in the sector $0 \leq \theta_z \leq \alpha_0 - \delta$, and split the proof into three cases:

$$(I) \ 0 \leq \theta_z \leq \frac{\pi}{2} - \delta_5,$$

$$(II) \ \frac{\pi}{2} - \delta_5 \leq \theta_z \leq \frac{\pi}{2} + \delta_5,$$

and

$$(III) \ \frac{\pi}{2} + \delta_5 \leq \theta_z \leq \alpha_0 - \delta,$$

where $\delta, \delta_5 < \frac{1}{3}(\alpha_0 - \frac{\pi}{2})$. In each of these three cases, using the asymptotics (2.3.1) of $\operatorname{Re} G$, we deform the original integration contour \mathcal{L}_α into a plus-admissible contour Γ_z that passes through the saddle point s_z . Then, Lemma 2.2.1 gives us the asymptotics of $\Gamma \cap D(s_z)$ which is always the main term, and in each of the three cases we will need to show that the integral over $\Gamma_z \setminus D(s_z)$ is negligible.

3.1 Case I: $0 \leq \theta_z < \frac{\pi}{2} - \delta_5$. We introduce the curve

$$\Gamma_z = -J_1 + J_2 + J_3,$$

where

$$J_1 = \{s = \rho e^{-i\theta_0} : \rho \geq \rho_z\},$$

$$J_2 = \{s = \rho_z e^{i\theta} : -\theta_0 \leq \theta \leq \theta_0\},$$

$$J_3 = \{s = \rho e^{i\theta_0} : \rho \geq \rho_z\},$$

with $\theta_0 = \frac{\pi}{2} + \delta_4$, and then split the arc J_2 into three parts,

$$J_2 = J_+ \cup J'_2 \cup J''_2,$$

where

$$J_+ = \{s \in J_2 : |s - s_z| \leq \rho_z^{1-\delta_1}\},$$

$$J'_2 = \{s \in J_2 : |s - s_z| \geq \rho_z^{1-\delta_1}, |\theta - \theta_z| \leq \delta_6\},$$

$$J''_2 = J_2 \setminus (J_+ \cup J'_2).$$

It is easy to see that the arc J_2 is plus-admissible, so it remains to show that the integrals

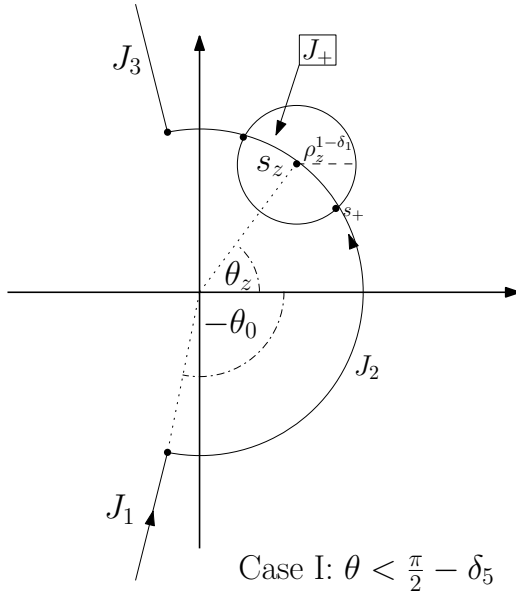


Fig. 2: Γ_z

over J'_2 , J''_2 , J_1 , and J_3 are negligible.

3.1.1 Integral over J'_2 . Here, we will use estimate (2.4.2) in Lemma 2.4.1. For $|\theta - \theta_z| \leq \delta_6$, we have $\delta_6 \leq \theta \leq \frac{\pi}{2} - \delta_5 + \delta_6$. Therefore,

$$\begin{aligned} h(\theta) &= -\cos \theta + (\theta - \theta_z) \sin \theta \leq -\cos\left(\frac{\pi}{2} - \delta_5 + \delta_6\right) + \delta_6 \\ &\leq -\frac{\delta_5 - \delta_6}{\pi} + \delta_6 \leq -c < 0, \end{aligned}$$

provided that $\delta_6 \leq \frac{1}{2}\delta_5$. Hence, in this range,

$$\frac{\partial^2}{\partial \theta^2} \operatorname{Re} G(z, \rho_z e^{i\theta}) \leq -c\varepsilon(\rho_z)\rho_z,$$

and then

$$\operatorname{Re} G(z, \rho_z e^{i\theta}) \leq \operatorname{Re} G(z, s_z) - c\varepsilon(\rho_z)\rho_z(\theta - \theta_z)^2.$$

Thus,

$$\begin{aligned} \left| \int_{J'_2} e^{G(z,s)} ds \right| &\leq e^{\operatorname{Re} G(z,s_z)} \rho_z \int_{c\rho_z^{-\delta_1} \leq |\theta - \theta_z| \leq \delta_6} e^{-c\varepsilon(\rho_z)\rho_z(\theta - \theta_z)^2} d\theta \\ &\leq C e^{\operatorname{Re} G(z,s_z)} \sqrt{\frac{\rho_z}{\varepsilon(\rho_z)}} \cdot e^{-c(\varepsilon(\rho_z)\rho_z^{1-\delta_1})^2} \end{aligned}$$

with negligible RHS.

3.1.2 Integral over J''_2 . Now we will use estimate (2.4.1) in Lemma 2.4.1. We claim that the function $f(\theta, \theta_z) = \cos \theta - \cos \theta_z + (\theta - \theta_z) \sin \theta$, which appears on the RHS of (2.4.1) is strictly positive whenever $|\theta| \leq \frac{\pi}{2} + \delta_4$, $0 \leq \theta_z \leq \frac{\pi}{2} - \delta_5$. Indeed, since $f'_\theta(\theta, \theta_z) = (\theta - \theta_z) \cos \theta$, the function $\theta \mapsto f(\theta, \theta_z)$ has a zero local minimum at $\theta = \theta_z$ and two positive local maxima at $\theta = \pm \frac{\pi}{2}$. Hence, it suffices to check that the values $f(\frac{\pi}{2} + \delta_4, \theta_z)$ and $f(-\frac{\pi}{2} - \delta_4, \theta_z)$ are positive. Since $f'_{\theta_z}(\theta, \theta_z) = -2 \sin \theta$ is negative at $\theta = \frac{\pi}{2} + \delta_4$ and positive at $\theta = -\frac{\pi}{2} - \delta_4$, we see that $f(\frac{\pi}{2} + \delta_4, \theta_z) \geq f(\frac{\pi}{2} + \delta_4, \frac{\pi}{2} - \delta_5)$ and $f(-\frac{\pi}{2} - \delta_4, \theta_z) \geq f(-\frac{\pi}{2} - \delta_4, 0)$. Finally, expanding in δ_4 and δ_5 , we get

$$\begin{aligned} f\left(\frac{\pi}{2} + \delta_4, \frac{\pi}{2} - \delta_5\right) &= -\sin \delta_4 - \sin \delta_5 + (\delta_4 + \delta_5) \cos \delta_4 \\ &= \frac{1}{6}(\delta_4^3 + \delta_5^3) - \frac{1}{2}\delta_4^2(\delta_4 + \delta_5) + O(\delta_4^4 + \delta_5^4) \\ &= \frac{1}{2}(\delta_4 + \delta_5) \cdot (\delta_5^2 - \delta_4\delta_5 + \frac{1}{2}\delta_4^2) + O(\delta_4^4 + \delta_5^4) > 0, \end{aligned}$$

$\delta_4 \leq \frac{1}{2}\delta_5$, and

$$f\left(-\frac{\pi}{2} - \delta_4, 0\right) = -\sin \delta_4 - 1 + \left(\frac{\pi}{2} + \delta_4\right) \cos \delta_4$$

$$= \frac{\pi}{2} - 1 + O(\delta_4) > 0.$$

This proves the claim, which immediately yields that on J''_2 we have $\operatorname{Re} G(z, s) \leq G(z, s_z) - c\rho_z\varepsilon(\rho_z)$, and therefore, the integral over J''_2 is negligible.

3.1.3 Integrals over J_1 and J_3 . Suppose that $s \in J_3$, that is, $s = \rho e^{i\theta_0}$, $\rho \geq \rho_z$. Then, by Lemma 2.6.1,

$$\operatorname{Re} G(z, \rho e^{i\theta_0}) \leq \operatorname{Re} G(z, \rho_z e^{i\theta_0}) - c\varepsilon(\rho_z)(\rho - \rho_z).$$

Besides, we already know that

$$\operatorname{Re} G(z, \rho_z e^{i\theta_z}) \leq \operatorname{Re} G(z, \rho_z e^{i\theta_0}) - c_1\varepsilon(\rho_z)\rho_z.$$

Thus,

$$\operatorname{Re} G(z, \rho e^{i\theta_0}) \leq \operatorname{Re} G(z, \rho_z e^{i\theta_z}) - c_1\varepsilon(\rho_z)\rho_z - c\varepsilon(\rho_z)(\rho - \rho_z),$$

and therefore, the integral over J_3 is negligible. For the same reason, the integral over J_1 is negligible as well.

3.2 Case II: $\frac{\pi}{2} - \delta_5 \leq \theta_z \leq \frac{\pi}{2} + \delta_5$. We put, as in the previous case, $\theta_0 = \frac{\pi}{2} + \delta_4$ with $\delta_4 \geq 2\delta_5$. Consider the straight line $\{s = s_z + te^{i3\pi/4} : t \in \mathbb{R}\}$ and denote by s_+ its intersection point with the ray $\{\arg(s) = \theta_0\}$ and by $s_- \neq s_z$ its intersection point with the circumference $|s| = \rho_z$. Let J be the segment $[s_-, s_+]$. Clearly, it is a plus-admissible curve. Our integration contour will be

$$\Gamma_z = -J_1 + J_2 + J + J_3,$$

where

$$J_1 = \{s = \rho e^{-i\theta_0} : \rho \geq \rho_z\},$$

$$J_2 = \{s = \rho_z e^{i\theta} : -\theta_0 \leq \theta \leq \arg(s_-)\},$$

$$J_3 = \{s = \rho e^{i\theta_0} : \rho \geq |s_+|\}.$$

The main term in the asymptotics comes from the segment $J_+ = J \cap D(s_z)$, and we need to check that the four remaining integrals over J_1 , J_2 , $J \setminus J_+$, and J_3 are negligible. Estimates of the integrals over J_1 and J_3 follow the same lines as in 3.1.3. So here we estimate only the integrals over $J \setminus J_+$ and J_2 .

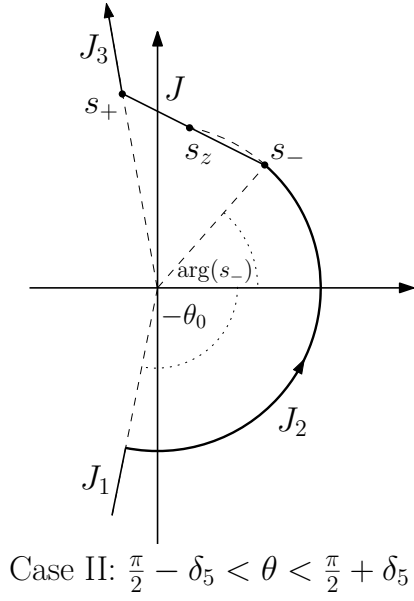
3.2.1 Integral over $J \setminus J_+$. Here, we use Lemma 2.5.1 with $\varphi = \frac{3\pi}{4}$. Since, for $s \in [s_-, s_+]$, $\cos(\frac{3\pi}{2} - \arg(s)) = -\sin(\arg s) \leq -c < 0$, Lemma 2.5.1 yields

$$\frac{\partial^2}{\partial t^2} \operatorname{Re} G(z, s_z + te^{i\frac{3\pi}{4}}) \leq -c\rho_z\varepsilon(\rho_z)$$

whenever $s = s_z + te^{i\frac{3\pi}{4}} \in J$, whence,

$$\operatorname{Re} G(z, s_z + te^{i\frac{3\pi}{4}}) \leq \operatorname{Re} G(z, s_z) - c\rho\varepsilon(\rho_z)t^2.$$

Since we are on $J \setminus J_+$, we integrate only over $|t| \geq c\rho_z^{1-\delta_1}$ and see that the integral over $J \setminus J_+$ is negligible.

Fig. 3: Γ_z

3.2.2 Integral over J_2 . By estimate (2.4.1), we have

$$\operatorname{Re} G(z, \rho_z e^{i\theta}) - \operatorname{Re} G(z, \rho_z e^{i\theta_z}) = -(f(\theta, \theta_z) + o(1))\rho_z \varepsilon(\rho_z),$$

with $f(\theta, \theta_z) = \cos \theta - \cos \theta_z + (\theta - \theta_z) \sin \theta$. The same elementary analysis as in 3.1.2 shows that, for $-\theta_0 \leq \theta \leq \theta_- = \arg(s_-)$ and $\frac{\pi}{2} - \delta_5 \leq \theta_z \leq \frac{\pi}{2} + \delta_5$, one has $f(\theta, \theta_z) \geq f(\theta_0, \theta_z) \geq c > 0$. This implies that the integral over the arc J_2 is negligible.

3.3 Case III: $\frac{\pi}{2} + \delta_5 \leq \theta_z \leq \alpha_0 - \delta$. Here, we take the contour

$$\Gamma_z = e^{i\theta_z} \mathbb{R}_+ - e^{-i\theta_z} \mathbb{R}_+ = J_1 - J_2.$$

The ray $J_1 = e^{i\theta_z} \mathbb{R}_+$ is plus-admissible and the main term in the asymptotics of the integral comes from integration over the segment $J_1 \cap D(s_z)$. Thus, we need to show that the integrals over $J_1 \setminus D(s_z)$ and $J_2 = e^{-i\theta_z} \mathbb{R}_+$ are negligible.

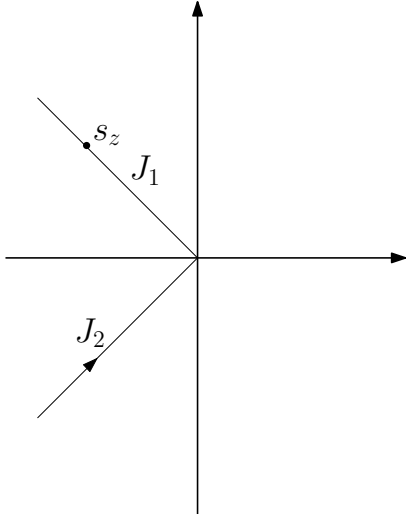
3.3.1 Integral over $J_1 \setminus D(s_z)$. We split $J_1 \setminus D(s_z)$ into four parts:

- (i) $0 \leq \rho \leq \rho_1$ (where ρ_1 is a large parameter that will be chosen later);
- (ii) $\rho_1 \leq \rho \leq \delta_3 \rho_z$;
- (iii) $\delta_3 \rho_z \leq \rho \leq \frac{3}{2} \rho_z$, $|\rho - \rho_z| \geq \rho_z^{1-\delta_1}$;

and

- (iv) $\rho \geq \frac{3}{2} \rho_z$.

By Lemma 2.7.1, the integral over $\rho \in [0, \rho_1]$ is negligible.



Case III: $\frac{\pi}{2} + \delta_5 < \theta_z < \alpha_0 - \delta$

Fig. 4: Γ_z

In the range $\rho_1 \leq \rho \leq \delta_3 \rho_z$, we have

$$\operatorname{Re} G(z, \rho e^{i\theta_z}) = \rho |\cos \theta_z| \int_{\rho}^{\rho_z} \frac{\varepsilon(u)}{u} du + O(\rho).$$

Consider the function

$$\ell(\rho) = \rho \int_{\rho}^{\rho_z} \frac{\varepsilon(u)}{u} du.$$

We have

$$\ell'(\rho) = \int_{\rho}^{\rho_z} \frac{\varepsilon(u)}{u} du - \varepsilon(\rho), \quad \ell''(\rho) = -\frac{\varepsilon(\rho)}{\rho} - \varepsilon'(\rho) = -(1 + o(1)) \frac{\varepsilon(\rho)}{\rho}.$$

If ρ_1 is chosen sufficiently large, then $\ell''(\rho)$ is negative on $[\rho_1, \delta_3 \rho_z]$. Hence, $\ell'(\rho)$ decreases. Since δ_3 is small and fixed, $\ell'(\delta_3 \rho_z) = (1 + o(1))(\log \frac{1}{\delta_3} - 1)\varepsilon(\rho_z) > 0$. Thus, $\ell(\rho)$ attains its maximal value at the end-point $\rho = \delta_3 \rho_z$ where it equals $(1 + o(1))\rho_z \varepsilon(\rho_z) \delta_3 \log \frac{1}{\delta_3}$. This shows that the integral over $[\rho_1, \delta_3 \rho_z]$ is negligible, provided that δ_3 is sufficiently small.

Now, consider the range $\delta_3 \rho_z \leq \rho \leq \frac{3}{2} \rho_z$, $|\rho - \rho_z| \geq \rho_z^{1-\delta_1}$. By Lemma 2.5.1, for $\delta_3 \rho_z \leq \rho \leq \frac{3}{2} \rho_z$ we have

$$\frac{\partial^2}{\partial \rho^2} \operatorname{Re} G(z, \rho e^{i\theta_z}) \leq -c \frac{\varepsilon(\rho_z)}{\rho_z},$$

whence

$$\operatorname{Re} G(z, \rho e^{i\theta_z}) \leq \operatorname{Re} G(z, s_z) - c \frac{\varepsilon(\rho_z)}{\rho_z} (\rho - \rho_z)^2.$$

Integrating this over $|\rho - \rho_z| \geq \rho_z^{1-\delta_1}$, we get a negligible expression.

The last range to consider is $\frac{3}{2} \rho_z \leq \rho < \infty$. Here, by Lemma 2.6.1,

$$\frac{\partial}{\partial \rho} \operatorname{Re} G(z, \rho e^{i\theta_z}) \leq -c \varepsilon(\rho_z).$$

Since we already know that

$$\operatorname{Re} G(z, \tfrac{3}{2}\rho_z e^{i\theta_z}) \leq \operatorname{Re} G(z, s_z) - c\varepsilon(\rho)z\rho_z,$$

we see that the integral over this range is also negligible.

3.3.2 Integral over J_2 . First, we note that

$$\operatorname{Re} G(z, \rho e^{i\theta_z}) - \operatorname{Re} G(z, \rho e^{-i\theta_z}) = (2 + o(1))\rho\varepsilon(\rho_z)\theta_z \sin \theta_z$$

uniformly in $z \in \bar{\Omega}(\alpha_0 - \delta)$, $z \rightarrow \infty$. Thus,

$$\operatorname{Re} G(z, \rho e^{-i\theta_z}) \leq \operatorname{Re} G(z, \rho e^{i\theta_z}) - c\rho\varepsilon(\rho_z).$$

Combining this observation with the estimates of $\operatorname{Re} G(z, \rho e^{i\theta_z})$ from 3.3.1, we readily conclude that the integral over J_2 is negligible as well.

4 Proof of Theorem 2

Throughout this section we fix an admissible function $\gamma(s) = L(s)^s$, which is analytic in the angle $|\arg(s + c_\gamma)| > 0$ (with $c_\gamma > 0$) and satisfies conditions (A), (B), and (C), and assume that

$$\bar{\varepsilon} = \limsup_{\rho \rightarrow \infty} \varepsilon(\rho) < 2. \quad (4.0.1)$$

We fix positive parameters σ_0 and δ_0 such that

$$0 < \sigma_0 < \min(c_\gamma, 1), \quad 0 < \delta_0 < \pi\left(\frac{1}{\bar{\varepsilon}} - \frac{1}{2}\right),$$

and recall that

$$zE(z) + \frac{1}{\gamma(0)} = \sum_{n \geq 0} \frac{z^n}{\gamma(n)}.$$

As before, we put $z = re^{i\psi}$ with $|\psi| \leq \pi$, and $s = \sigma + it = \rho e^{i\theta}$ with $|\theta| < \pi$.

We will also need the following elementary lower bound for the function γ :

Lemma 4.0.1. *For $s = \sigma + it$, $\sigma \geq -\sigma_0$, we have*

$$\frac{1}{|\gamma(s)|} \leq C_{\sigma_0} \frac{e^{a|t|}}{|\gamma(|\sigma|)|}$$

with any $a > \frac{\pi}{2} \bar{\varepsilon}$. In particular, this holds with some $a < \pi$.

Proof of Lemma 4.0.1: We have

$$\begin{aligned} \log |L(\sigma + it)| &= \int_0^{|\sigma + it|} \frac{\varepsilon(u)}{u} du + O(1) \\ &\geq \int_0^{|\sigma|} \frac{\varepsilon(u)}{u} du + O(1) = \log L(|\sigma|) + O(1), \end{aligned}$$

and

$$|t \arg L(\sigma + it)| \leq (|\arg(\sigma + it)| \varepsilon(\sigma + it)|t| + o(|t|)) \leq \frac{\pi}{2} a|t| + O(1),$$

with any $a > \bar{\varepsilon}$. This completes the proof of the lemma. \square

4.1 Applying the Abel-Plana summation. Our starting point is the representation

$$\sum_{n \geq 0} \frac{z^n}{\gamma(n)} = \int_{-\sigma_0}^{\infty} \frac{z^\sigma}{\gamma(s)} d\sigma - \frac{1}{2i} \int_{-\sigma_0}^{-\sigma_0+i\infty} \frac{z^s}{\gamma(s)} (\cot(\pi s) + i) ds + \frac{1}{2i} \int_{-\sigma_0}^{-\sigma_0-i\infty} \frac{z^s}{\gamma(s)} (\cot(\pi s) - i) ds. \quad (4.1.1)$$

This is one of the versions of the classical Abel-Plana summation formula. It holds for any function $F(s)$ holomorphic on $\{\operatorname{Re}(s) \geq -\sigma_0\}$ with convergent series $\sum_n F(n)$, which satisfies

$$\lim_{\sigma \rightarrow +\infty} |F(\sigma + it)| e^{-b|t|} = 0$$

with some $b < 2\pi$, uniformly in $t \in \mathbb{R}$. For $F(s) = z^s/\gamma(s)$, the latter condition immediately follows from Lemma 4.0.1.

Note that the LHS of (4.1.1) is an entire function of z , while the integrals on the RHS are analytic functions in the cut plane $|\arg(z)| < \pi$ with continuous boundary values on the upper and lower banks of the cut $\arg(z) = \pm\pi$.

4.2 Estimating the integrals over vertical lines. Here we show that both integrals over vertical lines on the RHS of (4.1.1) are $o(1)$ uniformly in $\arg(z)$ when $z \rightarrow \infty$, and therefore can be neglected. Since both estimates follow the same lines, we estimate only the 2nd integral on the RHS of (4.1.1).

Noting that

$$\cot(\pi s) + i = 2i \frac{e^{2\pi i s}}{e^{2\pi i s} - 1}$$

and recalling that $s = -\sigma_0 + it$ with $0 < \sigma_0 < 1$ and $t \geq 0$, we get

$$|\cot(\pi(-\sigma_0 + it)) + i| \leq C e^{-2\pi t}, \quad t \geq 0.$$

The rest follows from Lemma 4.0.1:

$$\left| \int_{-\sigma_0}^{-\sigma_0+i\infty} \frac{z^s}{\gamma(s)} (\cot(\pi s) + i) ds \right| \leq C_{\sigma_0} \frac{r^{-\sigma_0}}{\gamma(\sigma_0)} \int_0^\infty e^{(-\psi+a-2\pi)t} dt.$$

Since $\psi \geq -\pi$ and $a < \pi$, we are done.

4.3 Estimating the main integral. Thus,

$$\sum_{n \geq 0} \frac{z^n}{\gamma(n)} = \int_{-\sigma_0}^{\infty} \frac{z^\sigma}{\gamma(s)} d\sigma + o(1)$$

uniformly in $|\psi| \leq \pi$, and the proof of Theorem 2 boils down to estimation of the integral on the RHS.

Similarly to the proof of Theorem 1, we split the proof into three cases:

$$(I) \ z \in \bar{\Omega}(\frac{\pi}{2} - \delta_0),$$

$$(II) \ z \in \bar{\Omega}(\frac{\pi}{2} + \delta_0) \setminus \Omega(\frac{\pi}{2} - \delta_0),$$

and

$$(III) \ z \in \mathbb{C} \setminus \Omega(\frac{\pi}{2} + \delta_0).$$

In the first two cases, the saddle point $s_z = \rho_z e^{i\theta}$ lies in the sectors $|\theta_z| \leq \frac{\pi}{2} - \delta_0$ and $|\theta_z - \frac{\pi}{2}| \leq \delta_0$, correspondingly. For $z \in \bar{\Omega}(\frac{\pi}{2} - \delta_0)$, as in the proof of Theorem 1, the main term comes from integration over a neighbourhood of the saddle point and is given by Lemma 2.2.1. For z close to the boundary of $\Omega(\frac{\pi}{2})$, the contributions of the saddle point and of the neighbourhood of the starting point $s = -\sigma_0$ might be of the same order of magnitude. In the third case, the main term comes from integration over a neighbourhood of the starting point $s = -\sigma_0$.

As in the proof of Theorem 1, we assume that $\operatorname{Re}(z) \geq 0$, that is, $0 \leq \psi = \arg(z) \leq \pi$, and (in cases (I) and (II)) $0 \leq \theta_z \leq \alpha_0 - \delta$.

As above, we use the notation $G(z, s) = \log \gamma(s) - s \log z$. By δ, δ_i we denote small positive parameters that remain fixed during our estimates. Most of these parameters have been already defined in Section 2.

In the course of the proof of Theorem 2 all expressions that are

$$o(1) \sqrt{\frac{\rho_z}{\varepsilon(\rho_z)}} e^{-\operatorname{Re} G(z, s_z)} + o(1)$$

will be called *negligible*.

4.4 Case I: $z \in \bar{\Omega}(\frac{\pi}{2} - \delta)$. In this case, $0 \leq \theta_z \leq \frac{\pi}{2} - \delta_0$, and we will deform the integration contour to

$$\Gamma_z = [-\sigma_0, 0] + e^{i\theta_z} \mathbb{R}_+.$$

By Lemma 4.0.1, for $\operatorname{Re}(s) \geq 0$ and $\operatorname{Re}(z) \geq 0$, we have

$$\left| \frac{z^s}{\gamma(s)} \right| \leq \frac{C r^\sigma}{\gamma(\sigma)}.$$

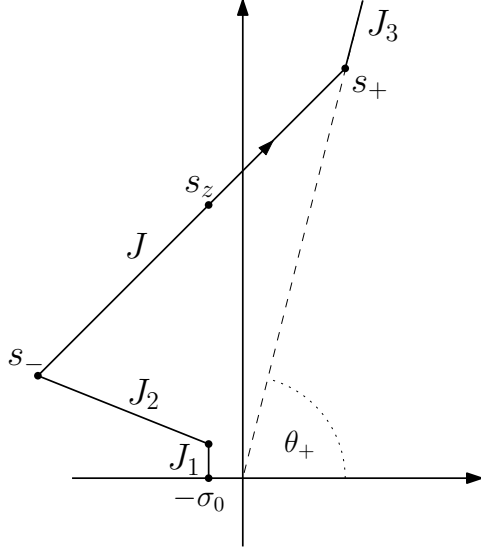
For $\sigma \rightarrow \infty$ the LHS converges to 0 faster than exponentially. This justifies rotation of the integration contour.

The function $z^\sigma / \gamma(\sigma)$ remains bounded for $\sigma \in [-\sigma_0, 0]$ and $|z| \geq 1$. Since the main term of the asymptotics comes from the integration over $e^{i\theta_z} \mathbb{R}_+ \setminus D(s_z, \rho_z^{1-\delta_1})$ and grows very fast, we may discard the integration over the segment $[-\sigma_0, 0]$ and estimate only the integral over $e^{i\theta_z} \mathbb{R}_+ \setminus D(s_z, \rho_z^{1-\delta_1})$. The estimates we need are practically identical to the ones from 3.3.1. We will not repeat these estimates, only mentioning that therein we integrated $\exp[\operatorname{Re} G(z, \rho e^{i\theta_z})]$ along the ray with strictly negative $\cos(\arg(s))$, while now we integrate $\exp[-\operatorname{Re} G(z, \rho e^{i\theta_z})]$ along the ray with strictly positive $\cos(\arg(s))$.

4.5 Case II: $z \in \bar{\Omega}(\frac{\pi}{2} + \delta_0) \setminus \Omega(\frac{\pi}{2} - \delta_0)$. In this case, $|\theta_z - \frac{1}{2}\pi| \leq \delta_0$. We fix an arbitrary positive $\eta < 1/e$, and put

$$\theta_- = \alpha - \delta, \quad \rho_- = \eta\rho_z, \quad s_- = \rho_- e^{i\theta_-}, \quad s_+ = \rho_+ e^{i\theta_+} = 2s_z - s_-$$

(i.e., s_z is the center of the interval $[s_-, s_+]$), and choose t_0 so that $\arg(-\sigma_0 + it_0) = \theta_-$. Then, we deform the integration contour to the union of three segments and a ray



Case II: $z \in \bar{\Omega}(\frac{\pi}{2} + \delta_0) \setminus \Omega(\frac{\pi}{2} - \delta_0)$

Fig. 5: Γ_z

$$\Gamma_z = J_1 + J_2 + J + J_3$$

$$= [-\sigma_0, -\sigma_0 + it_0] + [-\sigma_0 + it_0, s_-] + [s_-, s_+] + e^{i\theta_+}[\rho_+, \infty).$$

We note that if the parameter δ_0 is sufficiently small, then $0 < \theta_+ < \min(\frac{\pi}{2}, \theta_z)$. This justifies the deformation of the contour.

Next, we note that the segment J traverses the saddle-point s_z in the direction $\arg(s - s_z) = \frac{\pi}{2} - c\eta$ provided that δ_0 is sufficiently small. To apply Lemma 2.2.1, we need to traverse the saddle point in the direction $|\arg(s - s_z) - \frac{1}{2}\theta_z| \leq \frac{\pi}{4} - \delta_2$; taking into account that $|\theta_z - \frac{1}{2}\pi| \leq \delta_0$, this means that $\frac{1}{2}\delta_0 + \delta_2 \leq \arg(s - s_z) \leq \frac{\pi}{2} - (\frac{1}{2}\delta_0 + \delta_2)$. If δ_0 and δ_2 are sufficiently small, the direction $\frac{\pi}{2} - c\eta$ lies within this range, and therefore, Lemma 2.2.1 is applicable. It tells us that the integral over $J \cap D(s_z, \rho_z^{1-\delta_1})$ equals

$$(1 + o(1)) \sqrt{\frac{2\pi s_z}{\varepsilon(s_z)}} \exp[s_z \varepsilon(s_z)].$$

It remains to see that the contributions of the integrals over $J \setminus D(s_z, \rho_z^{1-\delta_1})$, J_1 , J_2 , and J_3 are all negligible.

4.5.1 Integral over J_1 . For $s \in J_1$, we have $|z^s/\gamma(s)| \leq Cr^{-\sigma_0}$. Hence, the integral over J_1 is bounded by $Cr^{-\sigma_0}$ and can be neglected.

4.5.2 Integral over J_2 . As in the previous case, for any given $\rho_1 > \rho_0$ (independent of z) and for $s \in [\rho_0 e^{i\theta-}, \rho_1 e^{i\theta-}]$, we have $|z^s/\gamma(s)| \leq Cr^{-\sigma_0}$. Hence, integrating over J_2 , we can integrate only over $\rho_1 \leq \rho \leq \eta\rho_z$. Then, by (2.3.1),

$$-\operatorname{Re} G(z, \rho e^{i\theta-}) = -\rho |\cos \theta_-| \int_{\rho}^{\rho_z} \frac{\varepsilon(u)}{u} du + O(\rho)$$

uniformly in $z \in \bar{\Omega}(\alpha - \delta)$.

We claim that, *given positive A and $\eta < 1/e$, we have*

$$\rho \int_{\rho}^{\rho_z} \frac{\varepsilon(u)}{u} du > A \log \rho, \quad \rho_1 \leq \rho \leq \eta\rho_z,$$

provided that ρ_1 is sufficiently large. Indeed, consider the function

$$\ell(\rho) = \rho \int_{\rho}^{\rho_z} \frac{\varepsilon(u)}{u} du - A \log \rho.$$

We have

$$\ell'(\rho) = \int_{\rho}^{\rho_z} \frac{\varepsilon(u)}{u} du - \varepsilon(\rho) - \frac{A}{\rho},$$

and

$$\ell''(\rho) = -\frac{\varepsilon(\rho)}{\rho} - \varepsilon'(\rho) + \frac{A}{\rho^2} = -(1 + o(1)) \frac{\varepsilon(\rho)}{\rho} < 0,$$

whenever ρ_1 is sufficiently large. Therefore, the function $\ell'(\rho)$ decays on $[\rho_1, +\infty)$. Noting that

$$\ell'(\eta\rho_z) = (1 + o(1))\varepsilon(\rho_z)(\log \frac{1}{\eta} - 1)$$

and recalling that $\eta < 1/e$, we conclude that $\ell' > 0$ on $[\rho_1, \eta\rho_z]$, i.e., ℓ increases therein. Therefore,

$$\ell(\rho) \geq \ell(\rho_1) = \rho_1 \int_{\rho_1}^{\rho_z} \frac{\varepsilon(u)}{u} du - \frac{A}{\rho_1} > 0,$$

provided that z is large enough. This proves the claim.

This claim immediately yields that

$$-\operatorname{Re} G(z, \rho e^{i\theta-}) \leq -2 \log \rho, \quad \rho_1 \leq \rho \leq \eta\rho_z.$$

Hence, the integral we are estimating does not exceed

$$\int_{\rho_1}^{\infty} \frac{d\rho}{\rho^2} = \frac{1}{\rho_1}.$$

Since we can choose ρ_1 as large as we need, the integral over J_2 is negligible.

4.5.3 Integral over $J \setminus D(s_z, \rho_z^{1-\delta_1})$. For $s \in J$, we have $\arg(s - s_z) = \frac{\pi}{2} - c\eta$ and $\arg(s) \in (\theta_+, \theta_-)$. Therefore,

$$\pi - 2c\eta - \theta_+ \leq 2\arg(s - s_z) - \arg(s) \leq \pi - 2c\eta - \theta_+.$$

So, if η is chosen sufficiently small, $\cos(2\arg(s - s_z) - \arg(s)) \geq c > 0$ for every $s \in J$. Then, by Lemma 2.5.1, the function $\operatorname{Re} G(z, \cdot)$ is concave on J , and moreover

$$\frac{\partial^2}{\partial t^2} [-\operatorname{Re} G(z, s_z + te^{i\phi})] \leq -c\rho_z \varepsilon(\rho_z)$$

whenever $s_z + te^{i\phi} \in J$. The rest of the argument is the same as in 3.2.1 and we skip it.

4.5.4 Integral over J_3 . We skip the estimate since it follows the same lines as the one in 3.1.3.

4.6 Case III: $z \in \mathbb{C} \setminus \bar{\Omega}(\frac{\pi}{2} + \delta_0)$. In this case, the saddle-point s_z may not exist (more precisely, it may live on another sheet of the Riemann surface of $\log z$). Nevertheless, given $z = re^{i\psi}$, we define a positive value ρ_z by equation

$$\int_0^{\rho_z} \frac{\varepsilon(u)}{u} du = \log r.$$

Then, we choose $t_0 > 0$ so that $\arg(-\sigma_0 + it_0)$, put $\theta_0 = \frac{\pi + \delta_0}{2}$, $\rho_0 = |\sigma_0 + it_0|$, and deform

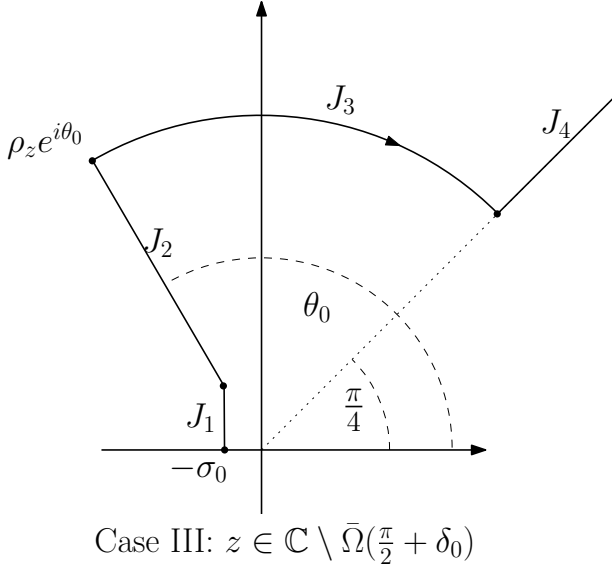


Fig. 6: Γ_z

the contour to

$$\Gamma_z = J_1 + J_2 - J_3 + J_4$$

$$= [-\sigma_0, -\sigma_0 + it_0] + [\rho_0 e^{i\theta_0}, \rho_z e^{i\theta_0}] - \{\rho_z e^{i\theta} : \frac{\pi}{4} \leq \theta \leq \theta_0\} + e^{i\pi/4} [\rho_z, +\infty).$$

As in the previous cases, the integral over J_1 is bounded by $Cr^{-\sigma_0}$, and we need to estimate the other three integrals.

4.6.1 Integral over J_2 . For sufficiently large $z \in \mathbb{C} \setminus \bar{\Omega}(\frac{\pi}{2} + \delta_0)$, we have

$$\psi = \arg(z) \geq \left(\frac{\pi}{2} + \frac{3}{4}\delta_0\right)\varepsilon(\rho_z),$$

while for $s \in J_2$, $\theta_0 = \arg(s) = \frac{\pi}{2} + \frac{1}{2}\delta_0$. Therefore,

$$-\operatorname{Re} G(z, \rho e^{i\theta_0}) \leq -\rho |\cos \theta_0| \int_{\rho}^{\rho_z} \frac{\varepsilon(u)}{u} du - \rho(\psi - \theta_0 \varepsilon(\rho)) \sin \theta_0 + \rho o(\varepsilon(\rho)) \quad (4.6.1)$$

$$\leq -\rho \int_{\rho}^{\rho_z} \left(\frac{\varepsilon(u)}{u} |\cos \theta_0| + \left(\frac{\pi}{2} + \frac{1}{2}\delta_0\right) \varepsilon'(u) \sin \theta_0 \right) du - \frac{1}{4}\delta_0 \rho \varepsilon(\rho_z) \sin \theta_0 + \rho o(\varepsilon(\rho))$$

$$\leq -c\delta_0 \rho \left[\int_{\rho}^{\rho_z} \frac{\varepsilon(u)}{u} du + \varepsilon(\rho_z) \right].$$

Then, as in the estimate of the similar integral in 4.5.2, we split the integral into three pieces: (a) $\rho_0 \leq \rho \leq \rho_1$ with sufficiently large ρ_1 , (b) $\rho_1 \leq \rho \leq \eta \rho_z$, and (c) $\eta \rho_z \leq \rho \leq \rho_z$.

(a) For $\rho_0 \leq \rho \leq \rho_1$, as before, we have $|z^s/\gamma(s)| \leq Cr^{-s_0}$, which does the job.

(b) Then, similarly to 4.5.2, we choose sufficiently large ρ_1 and η much smaller than δ_0 , so that, for $\rho_1 \leq \rho \leq \eta \rho_z$,

$$c\delta_0 \rho \int_{\rho}^{\rho_z} \frac{\varepsilon(u)}{u} du > 2 \log r.$$

This bounds the integral over the second piece by $1/\rho_1$.

(c) At last, on the third piece, we use that

$$-\operatorname{Re} G(z, \rho e^{i\theta_0}) \leq -c\delta_0 \rho \varepsilon(\rho_z) \leq -c\delta_0 \eta \rho_z \varepsilon(\rho_z),$$

which is, by far, more than we need.

4.6.2 Integral over J_3 . Using estimate (4.6.1), and recalling that

$$\psi \geq \left(\frac{\pi}{2} + \frac{3}{4}\delta_0\right)\varepsilon(\rho_z),$$

while $0 \leq \theta \leq \frac{\pi}{2} + \frac{3}{4}\delta_0$, we get

$$-\operatorname{Re} G(z, \rho_z e^{i\theta}) \leq -\rho_z(\psi - \theta \varepsilon(\rho_z)) + \rho_z o(\varepsilon(\rho_z)) \leq -c\delta_0 \rho_z \varepsilon(\rho_z),$$

which immediately yields that the integral over J_3 decays fast to zero.

4.6.3 Integral over J_4 . Here using once again estimate (4.6.1) and recalling that $\psi < -\frac{\pi}{2} \varepsilon(\rho_z)$, $\theta = \frac{\pi}{4}$, and $\rho \geq \rho_z$, we get

$$-\operatorname{Re} G(z, \rho e^{i\pi/4}) \leq -\frac{\rho}{\sqrt{2}} \int_{\rho_z}^{\rho} \left(\frac{\varepsilon(u)}{u} - \frac{\pi}{4} \varepsilon'(u) \right) du - \frac{\pi}{4} \frac{\rho}{\sqrt{2}} \varepsilon(\rho_z) + \rho \varepsilon(\rho)$$

$$\leq -c\rho \left[\int_{\rho_z}^{\rho} \frac{\varepsilon(u)}{u} du + \varepsilon(\rho_z) \right]$$

$$\leq -c\rho \varepsilon(\rho_z),$$

and then,

$$\int_{\rho_z}^{\infty} e^{-c\rho\varepsilon(\rho_z)} d\rho \leq \frac{C}{\varepsilon(\rho_z)} e^{-c\rho_z\varepsilon(\rho_z)},$$

which decays very fast to zero, uniformly in $z \in \mathbb{C} \setminus \bar{\Omega}(\frac{\pi}{2} + \delta_0)$, $z \rightarrow \infty$. This completes the proof of Theorem 2. \square

Appendix A Proof of Theorem 3

A.1 Slowly varying function. We will need several well-known proprieties of slowly-varying functions, which we summarize in the following lemma:

Lemma A.1.1. *Suppose that the function $\varepsilon \in C^1[0, \infty)$ is such that $\rho\varepsilon'(\rho) = o(\varepsilon(\rho))$, $\rho \rightarrow \infty$. Then,*

1. *For any interval $[a, b] \subset (0, \infty)$, $\lim_{\rho \rightarrow \infty} \sup_{\lambda \in [a, b]} \frac{\varepsilon(\lambda\rho)}{\varepsilon(\rho)} = 1$.*
2. *For any $\delta > 0$, the function $\rho \mapsto \rho^\delta \varepsilon(\rho)$ is eventually increasing and the function $\rho \mapsto \rho^{-\delta} \varepsilon(\rho)$ is eventually decreasing.*
3. *If I is an interval, $m \in C(\mathbb{R}_+ \times I)$ and $\delta > 0$, such that $\int_0^\infty t^{-\delta} |m(t, x)| dt < \infty$ and $\int_0^\infty t^\delta |m(t, x)| dt < \infty$, for all $x \in I$, then*

$$\lim_{\rho \rightarrow \infty} \int_0^\infty \frac{\varepsilon(\rho t)}{\varepsilon(\rho)} m(t, x) dt = \int_0^\infty m(t, x) dt$$

locally uniformly in I .

The proofs of assertions 1–3 can be found in [5], Lemmas/Theorems 1.3.1, 1.5.5 and 4.5.2 correspondingly.

A.2 Two lemmas that yield Theorem 3. We fix the function ℓ that satisfies assumptions of Theorem 3, that is, $\ell : [0, \infty) \rightarrow (0, \infty)$ is an unboundedly increasing C^1 -function such that the function

$$\rho \mapsto \rho \frac{\ell'(\rho)}{\ell(\rho)}$$

is slowly varying and bounded on $[0, \infty)$, and put

$$\gamma(s) = \exp \left(s^2 \int_0^\infty \frac{\ell'(u)}{\ell(u)} \cdot \frac{du}{s+u} \right), \quad |\arg(s)| < \pi, \quad (\text{A.2.1})$$

Then,

$$\log L(s) = \frac{1}{s} \log \gamma(s) = s \int_0^\infty \frac{\ell'(u)}{\ell(u)} \cdot \frac{du}{s+u}, \quad |\arg(s)| < \pi,$$

and

$$\varepsilon(s) = s \frac{L'(s)}{L(s)} = s \int_0^\infty u \frac{\ell'(u)}{\ell(u)} \frac{du}{(s+u)^2}, \quad |\arg(s)| < \pi.$$

The following two lemmas immediately yield Theorem 3.

Lemma A.2.1. *The function γ defined by (A.2.1) is admissible.*

Lemma A.2.2. *We have*

1. $\lim_{\rho \rightarrow \infty} \frac{\log L(\rho)}{\log \ell(\rho)} = 1.$
2. *If in addition there exists the limit $\lim_{\rho \rightarrow \infty} \rho \frac{\ell'(\rho)}{\ell(\rho)}$, then, $\lim_{\rho \rightarrow \infty} \frac{\ell(\rho)}{L(\rho)} = \ell(0).$*

A.3 Proof of Lemma A.2.1. By our assumption, $\frac{\ell'(u)}{\ell(u)} = O\left(\frac{1}{u}\right)$ as $u \rightarrow \infty$. Therefore, the integral in the definition of the function γ is absolutely and locally uniformly convergent in $\{s: |\arg(s)| < \pi\}$, and therefore, the function γ is analytic and non-vanishing therein. It is easy to see that positivity of ℓ' yields continuity of $\frac{1}{\gamma}(s)$ at $s = 0$.

By Lemma A.2.2, assertion 3, applied with

$$m(t, \theta) = \frac{e^{i\theta}}{(e^{i\theta} + t)^2}, \quad e^{i\theta} \int_0^\infty \frac{dt}{(e^{i\theta} + t)^2} = 1,$$

we get

$$\varepsilon(s) = (1 + o(1)) \rho \frac{\ell'(\rho)}{\ell(\rho)}, \quad s = \rho e^{i\theta}, \rho \rightarrow \infty, \quad (\text{A.3.1})$$

uniformly in any angle $|\arg(s)| \leq \pi - \delta$. This gives us the properties (A) and (C) in the definition of admissible functions.

In order to show that the property (B) also holds, we differentiate under the integral sign once again, and obtain

$$\rho \varepsilon'(\rho) = -\rho \int_0^\infty u \frac{\ell'(u)}{\ell(u)} \cdot \frac{\rho - u}{(u + \rho)^3} du.$$

Since,

$$\int_0^\infty \frac{\rho - u}{(\rho + u)^3} du = \int_0^\infty \frac{1 - t}{(1 + t)^3} dt = 0,$$

applying again Lemma A.2.2, this time with $m(t) = (1 - t)/(1 + t)^3$, we find that

$$\rho \varepsilon'(\rho) = o(1) \cdot \rho \frac{\ell'(\rho)}{\ell(\rho)}, \quad \rho \rightarrow \infty.$$

By (A.3.1), this gives us the property (B). □

A.4 Proof of Lemma A.2.2.

A.4.1 Proof of Part 1. Integration by part yields

$$\begin{aligned} \log L(\rho) &= \rho \int_0^\infty \frac{\ell'(u)}{\ell(u)} \cdot \frac{du}{\rho + u} = \rho \int_0^\infty \log \frac{\ell(u)}{\ell(0)} \cdot \frac{du}{(\rho + u)^2} \\ &= \int_0^\infty \log \frac{\ell(\rho t)}{\ell(0)} \cdot \frac{dt}{(1 + t)^2}. \end{aligned} \quad (\text{A.4.1})$$

Since

$$\lim_{u \rightarrow \infty} u \frac{(\log \ell(u))'}{\log \ell(u)} = \lim_{u \rightarrow \infty} u \frac{\ell'(u)}{\ell(u)} \cdot \frac{1}{\log \ell(u)} = 0,$$

the function $u \mapsto \log(\ell(u)/\ell(0))$ is slowly varying. Therefore, by Lemma A.1, assertion 3, applied with $m(t) = (1+t)^{-2}$, we get

$$\lim_{\rho \rightarrow \infty} \frac{\log L(\rho)}{\log \ell(\rho) - \log \ell(0)} = \int_0^\infty \frac{dt}{(1+t)^2} = 1.$$

Taking into account that $\lim_{\rho \rightarrow \infty} \ell(\rho) = \infty$, we conclude assertion 1.

A.4.2 Part 2. By (A.4.1),

$$\begin{aligned} \log L(\rho) - \log \frac{\ell(\rho)}{\ell(0)} &= \int_0^\infty \log \frac{\ell(\rho t)}{\ell(0)} \cdot \frac{dt}{(1+t)^2} - \log \frac{\ell(\rho)}{\ell(0)} \int_0^\infty \frac{du}{(1+t)^2} \\ &= \int_0^\infty \log \frac{\ell(\rho t)}{\ell(\rho)} \cdot \frac{1}{(1+t)^2} dt. \end{aligned}$$

We need to show that the integral on the RHS tends to 0 as $\rho \rightarrow \infty$.

We fix $\lambda > 1$, split the integral into three parts

$$\int_0^\infty \log \frac{\ell(\rho t)}{\ell(\rho)} \cdot \frac{1}{(1+t)^2} du = \int_0^{\lambda^{-1}} + \int_{\lambda^{-1}}^\lambda + \int_\lambda^\infty = I + II + III,$$

and estimate each of the three integrals separately.

By our assumption, $\frac{\ell'(u)}{\ell(u)} = O\left(\frac{1}{u}\right)$ as $u \rightarrow \infty$. Hence,

$$\left| \log \frac{\ell(\rho t)}{\ell(\rho)} \right| = \left| \int_\rho^{\rho t} \frac{\ell'(u)}{\ell(u)} du \right| \leq C |\log t|.$$

Therefore,

$$|I| \leq C \int_0^{\lambda^{-1}} \log \frac{\ell(\rho)}{\ell(\rho t)} dt \leq C \int_0^{\lambda^{-1}} |\log t| dt \leq C \frac{\log \lambda}{\lambda}.$$

Similarly,

$$|III| \leq C \int_\lambda^\infty \frac{\log t}{(t+1)^2} dt \leq C \frac{\log \lambda}{\lambda}.$$

Letting $a = \lim_{\rho \rightarrow \infty} \rho \frac{\ell'(\rho)}{\ell(\rho)}$, we see that

$$\lim_{\rho \rightarrow \infty} \log \frac{\ell(t\rho)}{\ell(\rho)} = \lim_{\rho \rightarrow \infty} \int_\rho^{\rho t} \frac{\ell'(u)}{\ell(u)} du = a \log t.$$

uniformly in $t \in [\lambda^{-1}, \lambda]$. Therefore,

$$\lim_{\rho \rightarrow \infty} II = \lim_{\rho \rightarrow \infty} a \int_{\lambda^{-1}}^\lambda \frac{\log t}{(t+1)^2} dt = 0.$$

Letting $\lambda \rightarrow \infty$, we obtain

$$\lim_{\rho \rightarrow \infty} \left[\log L(\rho) - \log \frac{\ell(\rho)}{\ell(0)} \right] = 0.$$

This finishes the proof of Lemma A.2.2, and hence, of Theorem 3. \square

Appendix B Admissible functions of positive type

Following Beurling, we say that a function γ is *of positive type* if it can be represented in the form

$$\gamma(s) = \exp \left(A + Bs + (s - a)^2 \int_0^\infty \frac{d\mu(s)}{u + s} \right),$$

with real constants A, B and a , and with a non-decreasing function μ such that

$$\int_0^\infty \frac{d\mu(s)}{u + 1} < \infty.$$

B.1 Note that the functions $\Gamma(s)$ and $\log_k^{\beta s}(s + c)$, $\beta > 0$, $c > 0$ and sufficiently big, are of positive type (here, as before, \log_k is the k th iterate of the logarithmic function). In the first case, this follows from the classical representation

$$\log \Gamma(s) = \log s - as + s^2 \int_0^\infty \frac{[u]}{u^2} \frac{du}{u + s},$$

where $[u]$ denotes the integer part of u . To see this in the second case, we put

$$F(s) = \frac{\log_k^b(s + c) - \log_k^b(c)}{s}, \quad |\arg z| < \pi$$

and apply the Cauchy formula

$$F(s) = \frac{1}{2\pi i} \int_{\Gamma_{\delta,R}} \frac{F(w)dw}{w - s}, \quad 0 < \delta \ll 1, \quad R \gg 1,$$

where the contour $\Gamma_{\delta,R}$ is defined in Figure 2. Then, letting $\delta \rightarrow 0$ and $R \rightarrow \infty$, we obtain

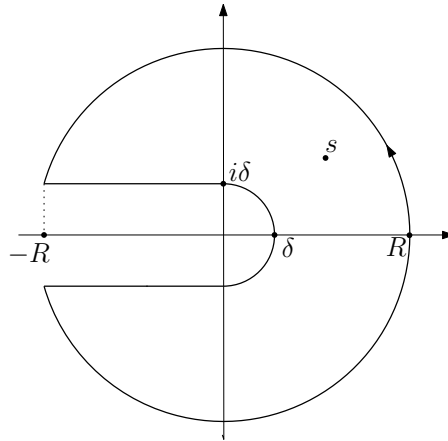


Fig. 7: Contour $\Gamma_{\delta,R}$

the representation

$$\log_k^b(s + c) = \log_k^b(c) + s \int_0^\infty \frac{\lambda(u)}{u} \frac{du}{u + s},$$

with

$$\lambda(u) = \frac{1}{2\pi i} \lim_{\epsilon \downarrow 0} [\log_k^b(-u + c + i\epsilon) - \log_k^b(-u + c - i\epsilon)].$$

If the constant c is big enough, the function $\log_k^b(u + c)$ is real on the positive half-line. Hence, by the Schwarz reflection principle, its jump on the negative ray is purely imaginary. Then, it is not difficult to see that $\lambda(u)$ is positive, provided that the constant c is big.

B.2 It is easy to check that functions constructed from these two examples using the rules described in 1.5.1, 1.5.4 and 1.5.5 are also functions of positive type.

B.3 We finish this discussion with the statement of Beurling's theorem [3, 4]:

Theorem (Beurling). *Every function γ of positive type is Mellin-positive definite, that is, represented by an absolutely convergent Stieltjes integral*

$$\gamma(s) = \int_0^\infty t^{s-1} d\nu(t) \quad (\text{B.3.1})$$

with a non-decreasing function ν on $[0, \infty)$.

It worth mentioning that this theorem can be also deduced from results presented in Berg's work [1, 2].

Note that, up to a normalization, the function ν in (B.3.1) is unique and can be recovered from γ by the inverse Mellin transform [14, §6.9]:

$$\nu(t) = \lim_{T \rightarrow +\infty} \frac{i}{2\pi} \int_{c-iT}^{c+iT} \gamma(s) t^{-s} \frac{ds}{s}, \quad c > 0.$$

Juxtaposing this with the representation (1.1.3) of the function K , we conclude that $K = \nu'$.

B.4 Finally, it worth mentioning that, the function K is the unique solution to the Stieltjes moment problem with the moments $(\gamma(n+1))_{n \geq 0}$ whenever

$$\limsup_{\rho \rightarrow \infty} \varepsilon(\rho) < 2. \quad (\text{B.4.1})$$

where, as before, $\varepsilon(\rho) = \rho \frac{L'(\rho)}{L(\rho)}$, in particular, whenever $\varepsilon(\rho) \rightarrow 0$ as $\rho \rightarrow \infty$. This immediately follows from the Carleman sufficient condition of determinacy: condition (B.4.1) readily yields that $L(\rho) = O(\rho^c)$, $\rho \rightarrow \infty$ with some $c < 2$, which, in turn, yields divergence of the series

$$\sum_{n \geq 0} \gamma(n+1)^{-1/(2n)} \geq \sum_{n \geq 1} \frac{1}{\sqrt{L(n)}} = +\infty.$$

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